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Longitudinal Vibration Analysis of Single-Walled Carbon Nanobeams

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ABSTRACT

In this study, the free longitudinal vibrations of single-walled carbon nanobeams are investigated using Eringen's nonlocal elasticity theory. The governing differential equation for longitudinal motion under clamped-clamped boundary conditions is derived and solved analytically using Mathematica. To validate the developed analytical formulation, a fully three-dimensional finite element model is implemented in COMSOL Multiphysics. The fundamental natural frequency obtained from the analytical solution compared with numerical simulation indicates a relative error of less than 0.15%, which confirms the high accuracy of the modeling. Furthermore, the results of the parametric analyses demonstrated that increasing the nonlocal scale parameter leads to a reduction in natural frequencies across all modes, representing the stiffness softening effect. This effect is significantly more pronounced in higher vibrational modes, such that by increasing the nonlocal parameter, the frequency drop of the third mode reaches more than 42%. Finally, examining the aspect ratio showed that small-scale effects completely dominate in short nanobeams, and with an increasing length-to-diameter ratio, the system behavior approaches classical elasticity.

1. Introduction

In recent years, single-walled carbon nanotubes (SWCNTs) and nanobeams have drawn significant attention in the field of nano-electromechanical systems (NEMS), nanosensors, and ultra-high-frequency resonators due to their exceptional mechanical properties, high stiffness-to-weight ratio, and extraordinary thermal stability[1]. Understanding the dynamic behavior and natural frequency response of these nanostructures is critical for the design and optimization of advanced nanodevices[2].

When analyzing structures at the nanoscale, classical continuum mechanics theories cannot provide accurate predictions because they ignore small-scale dimensional effects and internal characteristic length[3]. To bridge the gap between atomic simulations and classical continuum formulations without imposing heavy computational costs, nonlocal continuum mechanics has emerged as an efficient theoretical framework. Eringen's nonlocal elasticity theory assumes that the stress at a reference point in the continuum depends not only on the strain at that same point, but also on the strain field throughout the entire domain of the structure[4].

Recent studies in structural mechanics have extensively applied nonlocal elasticity theory to investigate bending, torsional, and longitudinal vibrations in various nanoscale structures[5].

Specifically, longitudinal wave propagation and axial vibration dynamics play a key role in acoustic wave sensors and high-frequency actuators. The incorporation of the nonlocal scale parameter in the governing differential equations generally leads to the stiffness softening phenomenon, wherein the natural frequency decreases as nonlocal effects become more prominent[6].

Although the analytical responses obtained from symbolic computation software (such as Mathematica) provide a deep understanding of the physics of nonlocal waves, numerical validation using three-dimensional finite element analysis (FEA) remains essential to confirm boundary conditions and structural dynamics. The main objective of this paper is to present an integrated analytical and numerically validated framework for the longitudinal vibrations of single-walled carbon nanotubes. In this study, the analytical formulation derived from Eringen's nonlocal model in Mathematica is compared with a three-dimensional solid mechanics model in COMSOL Multiphysics, the stiffness softening phenomenon is analyzed, and the accuracy of continuum models for nanobeam vibration applications is evaluated.

Furthermore, a deeper understanding of the dynamic responses of nanobeams under longitudinal oscillations requires the simultaneous investigation of transient behaviors and frequency stability in both space and time domains. In recent research, analytical solution methods based on symbolic computation have been recognized as highly accurate tools for mapping three-dimensional space-time responses, because, unlike approximation methods, they avoid numerical discretization errors in the time domain. This is of high importance, especially in the evaluation of higher-order vibration modes that are more sensitive to nonlocal scale parameters.

On the other hand, despite the capability of 1D and 2D analytical methods in predicting overall structural behavior, real systems at the nanoscale possess geometric and three-dimensional complexities. Therefore, utilizing three-dimensional finite element simulations (3D FEA) in advanced software such as COMSOL Multiphysics allows for a more accurate investigation of cylindrical geometry effects, three-dimensional mass distribution, and realistic support conditions. The integration of nonlocal analytical solution with 3D numerical simulation not only validates the theoretical formulations, but also establishes a reliable framework for the high-precision design of nanoscale components.

2-Literature Review

In recent years, nanostructures, particularly nanobeams and carbon nanotubes (CNTs), have attracted widespread attention in the design of nano-electromechanical systems (NEMS), precision sensors, actuators, and resonators due to their exceptional mechanical, electrical, and thermal properties. At the nanoscale, the mechanical behavior of structures is strongly dependent on dimensions and influenced by small-scale effects; therefore, classical continuum theories are incapable of accurately predicting the dynamic and vibrational responses of nanostructures because they fail to account for position-dependency effects and interatomic interactions.

To overcome this limitation, non-classical and size-dependent continuum mechanics theories have been developed. Among these, Eringen's Nonlocal Elasticity Theory, Nonlocal Strain Gradient Theory (NSGT), Modified Couple Stress Theory (MCST), and Two-phase Integral Models are considered among the most widely used frameworks for the precise analysis of nanostructure behavior. In addition to scale effects, factors such as:

- Functionally Graded Materials (FG/AFG): For optimizing mass and stiffness distribution
- Porosity and Anisotropy: In the material structure
- Thermal and Viscoelastic Environments: For modeling damping and temperature effects
- Curved Geometries and Structural Imperfections: Such as cracks or fluid-induced instabilities

play a decisive role in altering natural frequencies and vibration modes. Furthermore, from a computational perspective, a wide range of solution methods—from analytical and numerical solutions (such as GDQM, NURBS-based IGA, and spectral methods) to modern data-driven and machine learning approaches (ANN)—have been employed to accurately simulate these behaviors.

In the following, to achieve a comprehensive and structured perspective, the most important and recent studies conducted in the field of modeling, free vibrations, and size-dependent behavior in nanobeams and carbon nanotubes are reviewed and thoroughly analyzed:

Recent studies have extensively explored the dynamics of functionally graded (FG) nanobeams under various conditions. For instance, Kadioglu and Yaylı[7], analyzed the axial vibrations of FG nanobeams using nonlocal elasticity coupled with a Kelvin-Voigt viscoelastic model, highlighting the interplay between damping and scale parameters. Similarly, Tang et al.[8]. investigated thermal and size-dependent effects on linear and nonlinear vibrations via a refined finite element formulation. However, while these works focus on viscoelastic or thermal responses using simplified 1D or integral frameworks, they do not simultaneously address the 3D continuum cross-

sectional geometry effects and exact higher-mode stiffness softening. To bridge this gap, the present study integrates rigorous nonlocal analytical formulations with 3D finite element benchmarking.

Furthermore, subsequent investigations extended to specialized geometries and environmental conditions. For instance, Tang et al.[9] examined curved FG nanobeams on elastic foundations using a two-phase local/nonlocal FE model, while Tayeb et al.[10], evaluated the vibration characteristics of porous FG nanobeams based on higher-order shear deformation theory (HSDT). Although these studies successfully captured foundation, curvature, and porosity effects, they predominantly relied on complex multi-variable or specialized structural formulations without addressing the fundamental axial/longitudinal dynamics under 3D continuum and flexible boundary constraints.

Advanced modeling techniques and thermal environments have also been targeted. Tang and Cheng[11] analyzed the linear/nonlinear thermal vibrations of FG microbeams using two-phase integral models combined with GDQM, whereas Tang et al., implemented NURBS-based isogeometric analysis (IGA) for axially functionally graded (AFG) nanobeams[12]. While these sophisticated numerical approaches effectively handle thermal loads and complex geometries, they often involve heavy computational overhead and lack a direct, exact analytical evaluation of fundamental longitudinal dynamics coupled with 3D continuum behaviors.

Furthermore, studies focusing on non-rigid boundaries and data-driven methods have emerged. Kadioglu and Yaylı[13], investigated viscoelastic FG nanobeams under non-rigid elastic boundaries using Fourier series and Stokes' transformation, evaluating damping and scale interactions. Concurrently, Al-Dhahidi et al. developed machine learning-based reduced-order models (ROMs) to predict natural frequencies of tapered AFG nanobeams[14]. Although these approaches address boundary flexibility and computational acceleration, they rely on 1D beam assumptions or black-box estimations, thereby overlooking the detailed 3D continuum deformation fields and exact multi-dimensional stiffness softening resolved herein.

Uncertainty quantification and complex multi-physics interactions have also been explored. Kutluay and Atıcı[15], evaluated the stochastic free vibrations of nanobeams under uncertain boundaries using Monte Carlo simulations, while Ramezani et al.[16] analyzed the stability and vibrations of fluid-conveying curved nanotubes under thermal and magnetic fields. While these studies provide valuable insights into probabilistic variations and fluid-structure interactions, they typically operate within standard 1D beam frameworks, lacking a comprehensive 3D continuum mechanical benchmark for longitudinal dynamics.

Furthermore, advanced continuum shell theories and multi-parameter nonlocal models have been utilized. Strozzi et al.[17] analyzed SWCNTs using Sanders-Koiter shell theory combined with nonlocal strain gradient models to capture anisotropy, while Ebrahimi et al.[18] developed space-time fractional viscoelastic models for wave propagation in FGM nanobeams. Although these complex shell and fractional frameworks offer deep multi-scale insights, their mathematical complexity often obscures the direct physical mechanisms governing fundamental longitudinal dynamics under flexible boundaries, a limitation addressed by the present 3D-validated analytical formulation.

Finally, refined gradient formulations and exact solutions for composite nanobeams have been established. Barretta et al.[19] introduced a variationally consistent nonlocal strain gradient Timoshenko model validated via MD simulations. More recently, Borjalilou et al.[20] provided exact solutions for FG-CNTRC nanobeams using nonlocal Timoshenko theory, noting boundary-

dependent scale behaviors (e.g., stiffness hardening in cantilevers). Nevertheless, despite their analytical elegance, works such as Refs.[7], [13], and [20] remain restricted to 1D beam approximations. They do not account for full 3D continuum deformation fields, cross-sectional geometry constraints, or exact higher-mode stiffness softening under flexible boundaries—gaps that are systematically addressed in the present study via a coupled analytical-3D FEA framework.

3- Governing Equations

3-1 Displacement and Strain in Longitudinal Vibrations

In axial/longitudinal vibration, the displacement field has only one component along the axis, and transverse displacements are assumed to be zero:

$$u_x(x,t) = u(x,t), \quad u_y(x,t) = 0, \quad u_z(x,t) = 0 \qquad (1)$$

The linear longitudinal strain (classical) is given by:

$$\varepsilon_x x = \frac{\partial u(x,t)}{\partial x} \qquad (2)$$

3-2 Eringen's Nonlocal Constitutive Relation

According to Eringen's nonlocal elasticity theory, the nonlocal stress at a point depends not only on the strain at that same point, but also on the strain at all points in the domain. The differential form of this relation for one-dimensional longitudinal behavior is given by:

$$\sigma_{xx} - (e_0 a)^2 \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E\varepsilon_{xx} \qquad (3)$$

Where:

- σ_{xx}: nonlocal stress component
- E: Young's modulus of the material, which is variable for FG materials
- $e_0 a$: nonlocal parameter or internal length scale, where is a material constant and is the interatomic distance

Substituting the strain relation, the constitutive equation becomes as follows:

$$\left[1 - (e_0 a)^2 \frac{\partial^2}{\partial x^2}\right] \sigma_{xx} = E(x) \frac{\partial u}{\partial x} \qquad (4)$$

3-3 Derivation of Equations Using Hamilton's Principle

Hamilton's principle for a dynamical system state that:

$$\delta \int_{t_1}^{t_2} (T - U)\ dt = 0 \qquad (5)$$

a) Kinetic energy (T)

$$T = \frac{1}{2}\int_0^L m\,(x)\left(\frac{\partial u}{\partial t}\right)^2 dx \qquad (6)$$

where $m(x) = \rho(x)A(x)$ is the mass per unit length of the beam, ρ is the density, and A is the cross-sectional area.

b) Internal strain energy (U)

$$U = \frac{1}{2}\int_0^L \int_A \sigma_{xx}\,\varepsilon_{xx}\,dA\,dx = \frac{1}{2}\int_0^L N\,(x,t)\frac{\partial u}{\partial x}\,dx \qquad (7)$$

where $\mathrm{N(x,t)} = \int_{\mathrm{A}} \sigma_{\mathrm{xx}}\,\mathrm{dA}$ is the resultant local/axial force

By applying the variation symbol (δ) and integration by parts:

$$\int_{t_1}^{t_2}\int_0^L \left[-m(x)\frac{\partial^2 u}{\partial t^2} + \frac{\partial N}{\partial x}\right]\delta u\,dx\,dt + [N\delta u]_0^L = 0 \qquad (8)$$

Therefore, the dynamic equilibrium equation is obtained:

$$\frac{\partial N}{\partial x} = m(x)\frac{\partial^2 u}{\partial t^2} \qquad (9)$$

To rigorously validate the kinematic assumptions of the proposed 1D analytical model, a full 3D continuum solid mechanics model is developed in COMSOL Multiphysics under the classical limit ($\mathrm{e_0a} = 0$). Once the baseline 3D structural behavior is verified, the non-local length scale effects ($\mathrm{e_0a} > 0$) and stiffness softening phenomena are comprehensively analyzed via the exact analytical solver in Mathematica.

3-4 Final Governing Equation in Terms of Displacement

The nonlocal constitutive equation for axial dynamics is expressed as:

$$N - (e_0 a)^2\frac{\partial^2 N}{\partial x^2} = E(x)A(x)\frac{\partial u}{\partial x} \qquad (10)$$

To obtain the governing differential equation in terms of displacement $u(x,t)$, Eq. (10) is differentiated with respect to the spatial coordinate x:

$$\frac{\partial N}{\partial x} - (e_0 a)^2\frac{\partial^3 N}{\partial x^3} = \frac{\partial}{\partial x}\left[E(x)A(x)\frac{\partial u}{\partial x}\right]$$

Substituting the equation of dynamic equilibrium, $\frac{\partial N}{\partial x} = m(x)\frac{\partial^2 u}{\partial t^2}$, into the expression above yields the governing equation for non-homogeneous nanobeams:

$$\frac{\partial}{\partial x}\left[E(x)A(x)\frac{\partial u}{\partial x}\right] = m(x)\frac{\partial^2 u}{\partial t^2} - (e_0 a)^2 \frac{\partial^2}{\partial x^2}\left(m(x)\frac{\partial^2 u}{\partial t^2}\right) \qquad (11)$$

Special case: Homogeneous nanobeam with constant cross-section

If the material properties (E, ρ) and cross-sectional area (A) are constant along the length $(E(x) = E, m(x) = \rho A)$, Eq. (11) simplifies to the following explicit governing equation:

$$EA\frac{\partial^2 u}{\partial x^2} = \rho A\left[\frac{\partial^2 u}{\partial t^2} - (e_0 a)^2 \frac{\partial^4 u}{\partial x^2 \partial t^2}\right] \qquad (12)$$

This equation is the governing equation for the longitudinal free vibrations of the nonlocal nanobeam.

3-5 Boundary Conditions

At the two ends of the beam, $x = 0$ and $x = L$, one of the following two conditions must be satisfied for each end:

1. Clamped/Fixed:

$$u(x,t) = 0 \qquad (13)$$

2. Free*:*

$$N(x,t) = 0 \Longrightarrow EA\frac{\partial u}{\partial x} + \rho A(e_0 a)^2 \frac{\partial^3 u}{\partial x \partial t^2} = 0 \qquad (14)$$

3-6 Nondimensionalization of Equations

To generalize the analysis, the following dimensionless variables are defined:

$$X = \frac{x}{L}, \quad U = \frac{u}{L}, \quad \mu = \frac{e_0 a}{L}, \quad \tau = t\sqrt{\frac{E}{\rho L^2}} \qquad (15)$$

Dimensionless governing equation for the homogeneous nanobeam:

$$\frac{\partial^2 U}{\partial X^2} = \frac{\partial^2 U}{\partial \tau^2} - \mu^2 \frac{\partial^4 U}{\partial X^2 \partial \tau^2} \qquad (16)$$

Assuming a harmonic response of the form $U(X,\tau) = \bar{U}(X)e^{i\Omega\tau}$, where Ω is the dimensionless natural frequency*:*

$$\frac{d^2\bar{U}}{dX^2} + \frac{\Omega^2}{1-\mu^2\Omega^2}\bar{U} = 0 \qquad (17)$$

4- Numerical and analytical solution methods

In this section, the solution procedure for the governing differential equation of the longitudinal free vibrations of the nanobeam is presented. First, the exact analytical solution is developed to extract the closed-form expression of natural frequencies, and its implementation in Mathematica is described. Then, the finite element modeling (FEM) process in COMSOL Multiphysics will be explained to validate the results.

4-1 Exact analytical solution

The governing differential equation for the longitudinal free vibrations of a homogeneous nanobeam based on Eringen's nonlocal elasticity theory is given by:

$$EA\frac{\partial^2 u(x,t)}{\partial x^2} = \rho A\frac{\partial^2 u(x,t)}{\partial t^2} - \rho A(e_0 a)^2\frac{\partial^4 u(x,t)}{\partial x^2 \partial t^2}$$

To obtain the modal response, the longitudinal displacement is assumed using the separation of time and spatial variables (harmonic response):

$$u(x,t) = U(x)e^{i\omega t}$$

where U(x) is the longitudinal mode shape function and Ω is the natural frequency of vibration. Substituting this relation into the governing differential equation yields an ordinary differential equation (ODE):

$$EA\frac{d^2U(x)}{dx^2} + \rho A\omega^2 U(x) + \rho A(e_0 a)^2\omega^2\frac{d^2U(x)}{dx^2} = 0$$

By factoring out the second derivative$\frac{d^2U(x)}{dx^2}$, the equation simplifies to*:*

$$[EA + \rho A(e_0 a)^2\omega^2]\frac{d^2U(x)}{dx^2} + \rho A\omega^2 U(x) = 0$$

Or in standard form*:*

$$\frac{d^2U(x)}{dx^2} + \beta^2 U(x) = 0$$

where β the longitudinal wave number is given by*:*

$$\beta = \sqrt{\frac{\rho A\omega^2}{EA + \rho A(e_0 a)^2\omega^2}}$$

The general solution to this differential equation is:

$$U(x) = C_1\, cos(\beta x) + C_2\, sin(\beta x)$$

Application of the clamped-clamped boundary conditions*:*

For the clamped-clamped boundary conditions, the displacement must be zero at both ends :$U(0) = 0$ و $U(L) = 0$

1. From the condition $U(0) = 0$ it yields:$C_1 = 0$
2. From the condition $U(L) = 0$ it yields:$C_2 sin(\beta L) = 0$

For the non-trivial solution $C_2 \neq 0$, the characteristic equation is given by:

$$sin(\beta L) = 0 \Longrightarrow \beta L = n\pi \quad (n = 1,2,3, \dots)$$

By substituting the definition of βinto this relation and squaring both sides:

$$\frac{\rho A \omega_n^2}{EA + \rho A (e_0 a)^2 \omega_n^2} = \left(\frac{n\pi}{L}\right)^2$$

By solving this relation for Ω, the exact closed-form formula for the n-th longitudinal natural frequency of the nanobeam is obtained:

$$\omega_n = \frac{n\pi}{L} \sqrt{\frac{E/\rho}{1 + (e_0 a)^2 \left(\frac{n\pi}{L}\right)^2}}$$

Analytical note: This relation explicitly shows that the presence of the nonlocal parameter $(e_0 a)$in the denominator causes a reduction in the natural frequency $\omega_n$compared to the classical case $(e_0 a = 0)$, which corresponds to the stiffness softening phenomenon

4-2 Symbolic implementation and numerical solution in Mathematica

For numerical calculations, plotting the frequency spectra, and reconstructing the 3D space-time responses (U(x, t)), the above formulation has been implemented in the Mathematica environment:

1. Definition of the mechanical and geometric properties of the nanobeam as symbolic and numerical variables.
2. Solving the eigenvalue system of equations and extracting frequencies for various values of the nonlocal parameter.
3. Generation of the 3D response functions $\mathrm{U(x,t)} = U_n(x)\cos(\omega_n t)$ and plotting of 3D surface and modal graphs.

4-3 Finite Element Modeling in COMSOL Multiphysics

The 3D geometry of the cylindrical nanobeam with length $\mathrm{L} = 20\mathrm{nm}$ and cross-sectional radius $\mathrm{R} = 1\mathrm{nm}$ was meshed in COMSOL using dense 3D solid elements, as shown in **Figure 1**. To ensure the accuracy of calculations for higher modes, a mesh independence analysis was performed.

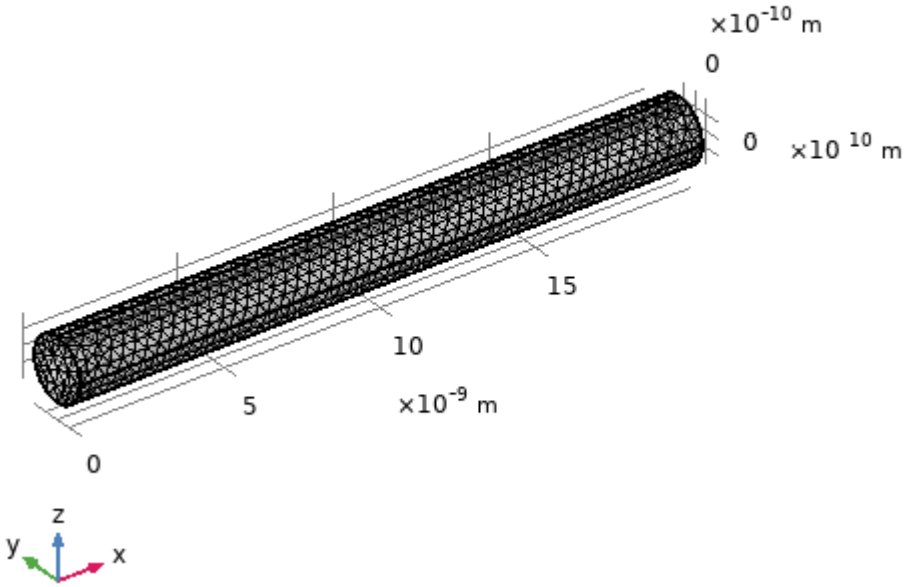


Figure 1. 3D finite element geometry and computational mesh of the cylindrical nanobeam in COMSOL Multiphysics.

To validate the analytical calculations, a 3D finite element model was developed in COMSOL Multiphysics (Solid Mechanics module):

1. **Geometry:** The nanobeam is modeled as a 3D cylinder with a length of $L = 20\text{nm}$ and a cross-sectional radius of $R = 1\text{nm}$.
2. **Meshing:** Dense volumetric elements (Free Tetrahedral/Sweep) were utilized to ensure numerical convergence and high accuracy in higher modes.
3. **Boundary Conditions:** A Fixed Constraint *was applied to the two selected cross-sections at* $x = 0$ *and* $x = L$.
4. **Study Type:** An **E**igenfrequency analysis was employed to extract the natural frequencies and longitudinal mode shapes.

4-4 Input Specifications and Reference Parameters

The geometric parameters and material properties used in all analytical calculations and simulations of this study are summarized in **Table 1**:

Table 1. Geometric parameters and reference mechanical properties of the carbon nanobeam

Young's modulus	1.0 TPa
Mass density	2300 kg/m^3
Length of the nanobeam	20 nm
Cross-sectional radius	1 nm
Range of variation of the nonlocal parameter	0.0 nm *to* 3.0 nm

5- Numerical Results and Discussion

In this section, first, the validity and accuracy of the presented analytical formulation are evaluated by comparing the natural frequencies obtained from the exact solution in Mathematica with the finite element simulation results in COMSOL Multiphysics. Next, the effects of the nonlocal scale parameter (e_0a) and the vibration mode number on the longitudinal natural frequencies and the space-time responses of the nanobeam are analyzed in detail.

5-1 Model Validation and Verification

To evaluate the accuracy of the analytical solution, the first-mode longitudinal natural frequency of the nanobeam under classical elasticity ($e_0a = 0$), calculated from the closed-form analytical formula, is compared with the 3D finite element solution output in COMSOL Multiphysics.

As shown in **Figure 2**, the longitudinal and cross-sectional displacement contours extracted from COMSOL were obtained at the fundamental frequency. The numerical comparison indicates:

- First-mode frequency (Mathematica analytical solution): $521.80 GHz$
- First-mode frequency (COMSOL finite element simulation): $521.04 GHz$
- Relative error: 0.146%

This excellent level of agreement and an error of less than 0.15% confirm the complete validity of the analytical formulation and the performed numerical simulation.

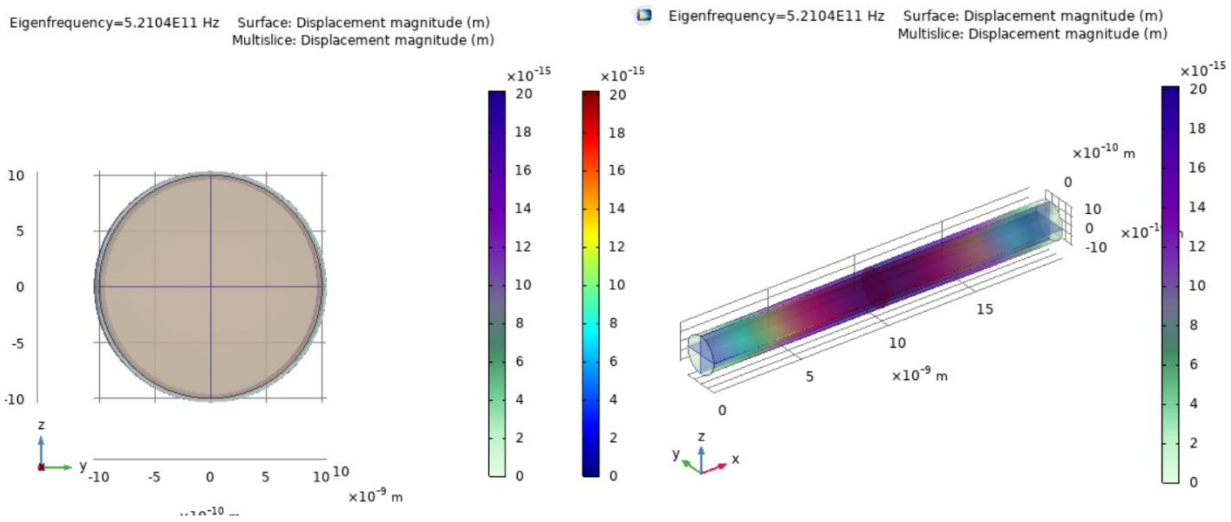


Figure 2. Finite element modal analysis in COMSOL at the fundamental natural frequency: (a) cross-sectional displacement field, and (b) 3D contour of the longitudinal displacement amplitude along the nanobeam.

5-2 Effect of the nonlocal scale parameter and the stiffness softening phenomenon

Figure 3 shows the variations of the longitudinal natural frequencies of the first, second, and third modes ($n = 1,2,3$) of the nanobeam as a function of the nonlocal parameter ($e_0 a$) ranging from $0.0nm$ to $3.0nm$.

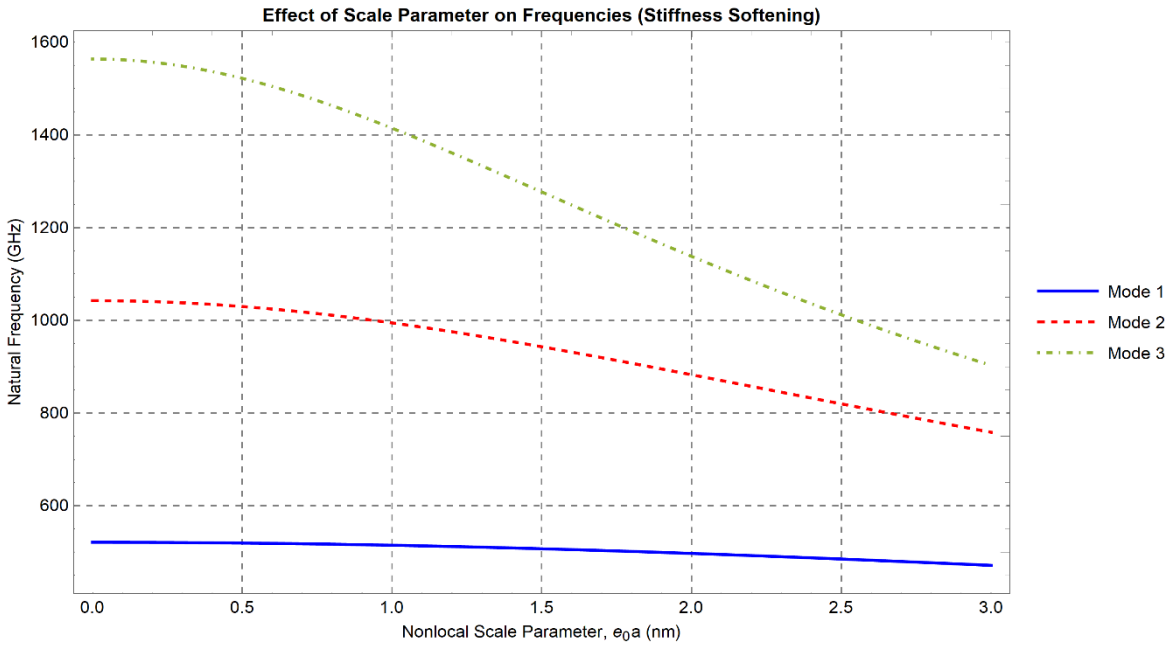


Figure 3. Variations of the first three longitudinal natural frequencies of the nanobeam versus the nonlocal scale parameter ($e_0 a$) and the illustration of the stiffness softening phenomenon.

Analysis of **Figure 3** reveals the following key points:

1. **Reduction of Natural Frequencies:** With an increase in the nonlocal parameter ($e_0 a$), the natural frequencies of all modes decrease monotonically. The physical cause of this phenomenon is structural stiffness softening; Eringen's nonlocal elasticity theory incorporates interatomic interaction effects, which effectively reduces the equivalent stiffness of the structure at the nanoscale.

2. **Sensitivity of Higher Modes:** The magnitude of the nonlocal effect is significantly more pronounced in higher vibration modes. For instance, as $e_0 a$ increases from$0.0nm$ to $3.0nm$

    - The frequency of the first mode ($n = 1$) decreases from $521.8GHz$ to approximately $470GHz$ (about a 10% reduction).
    - The frequency of the third mode ($n = 3$) drops from $1565.4GHz$ to roughly $900GHz$ (about a 42.5% reduction).

This demonstrates that higher modes, characterized by shorter wavelengths, exhibit a much higher sensitivity to small-scale effects.

5-3 Analysis of Space-Time Response and Longitudinal Mode Shapes

To fully understand the dynamical behavior of the nanobeam, the space-time response of the longitudinal displacement $u(x,t)$ and the three-dimensional modal shapes extracted from Mathematica are presented in **Figure 4** and **Figure 5**.

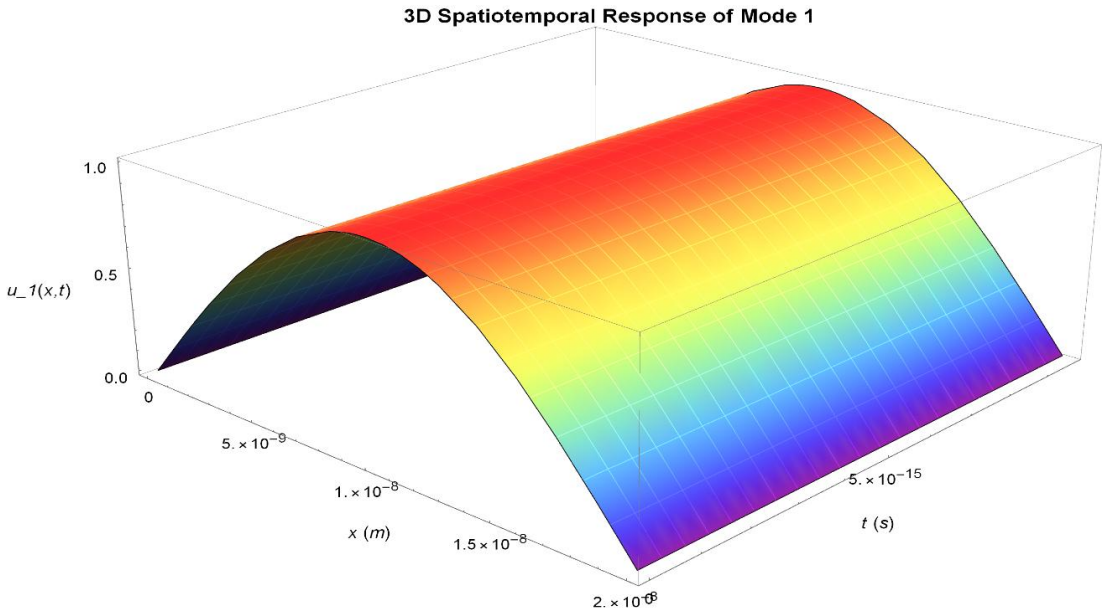


Figure 4. Three-dimensional space-time response of the longitudinal displacement u(x, t) for the first vibration mode (n = 1).

Figure 4 depicts the harmonic dynamic response u(x, t) along the nanobeam x over the time interval t, which indicates the stability and periodic behavior of the longitudinal vibrations.

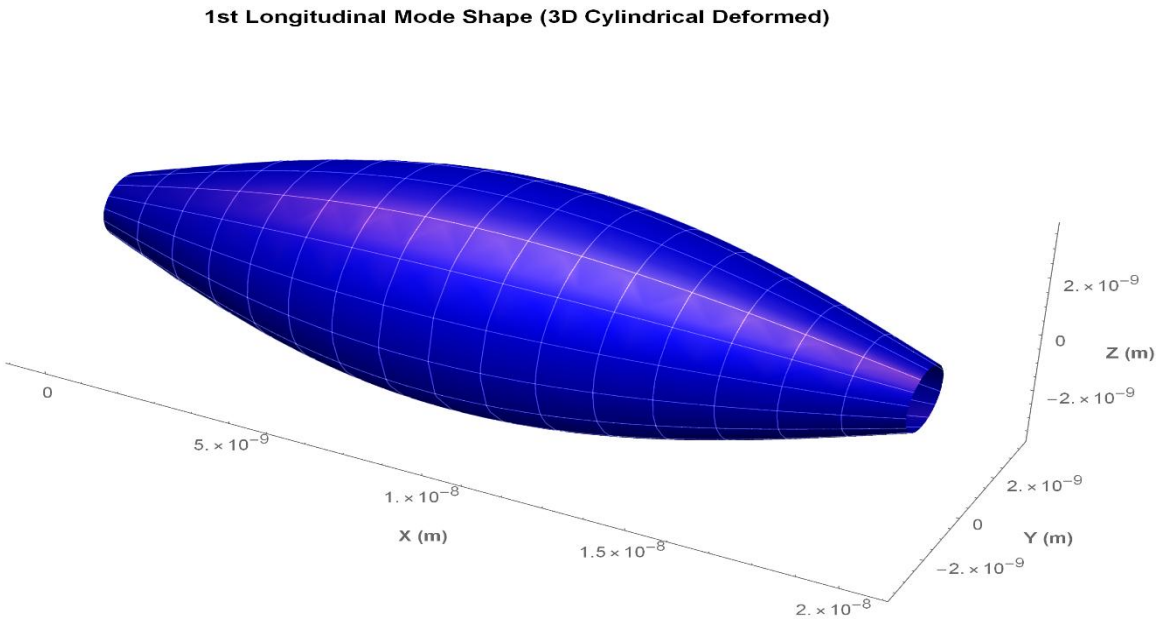


Figure 5. Three-dimensional cylindrical representation of the first modal deformation for the longitudinal vibrations of the nanobeam.

Figure 5 shows the three-dimensional cylindrical deformation of the nanobeam in the first vibration mode, where the maximum longitudinal displacement amplitude occurs at the mode antinode (the

middle section of the nanobeam). For a more precise comparison of the sensitivity of different modes to small-scale effects, the normalized frequency ratio (the difference between the nonlocal and classical frequencies) versus variations of the nonlocal parameter e_0a is evaluated in Figure 6.

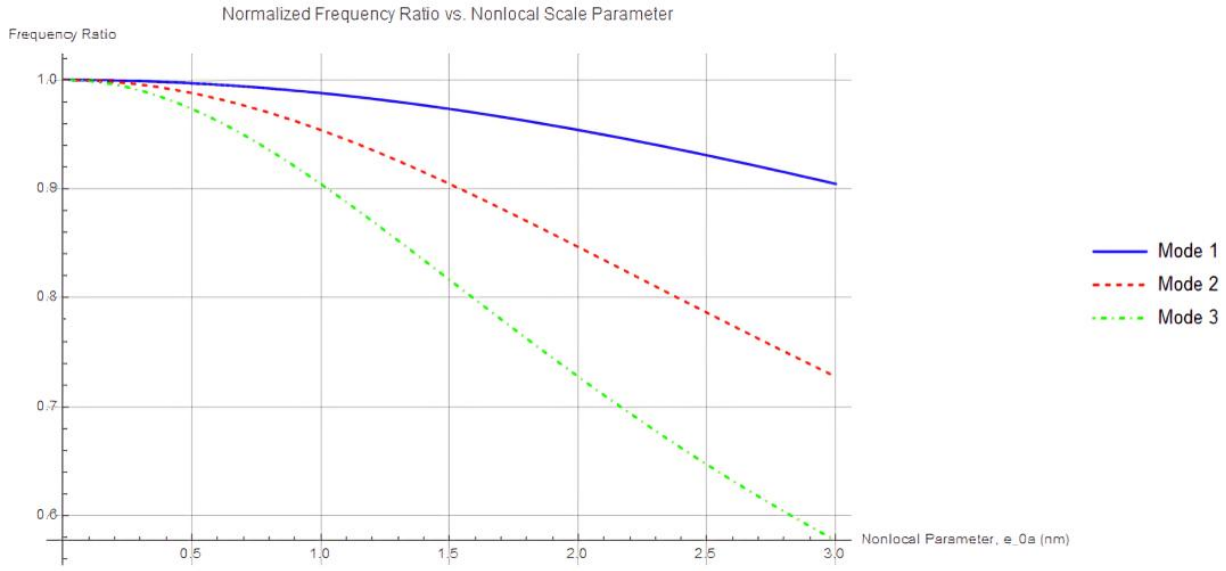


Figure 6. Variations of the normalized frequency ratio ($\frac{\omega_{nonlocal}}{\omega_{classical}}$) versus the nonlocal scale parameter (e_0a) for the first three modes of longitudinal vibrations.

According to Figure 6, in the classical case ($e_0a = 0$), the frequency ratio for all modes is equal to 1. As the nonlocal parameter increases up to 3 nm, the frequency ratio reaches approximately 0.9 (about a 10% reduction) for the first mode, about 0.73 (about a 27% reduction) for the second mode, and less than 0.58 (more than a 42% reduction) for the third mode. This proves that the stiffness softening effect is significantly more severe and decisive in higher vibration modes.

Figure 7 illustrates the variations of the frequency ratio versus the length-to-diameter ratio of the nanobeam (L/d) for a constant nonlocal parameter ($e_0a = 2$ nm) for the first and second modes.

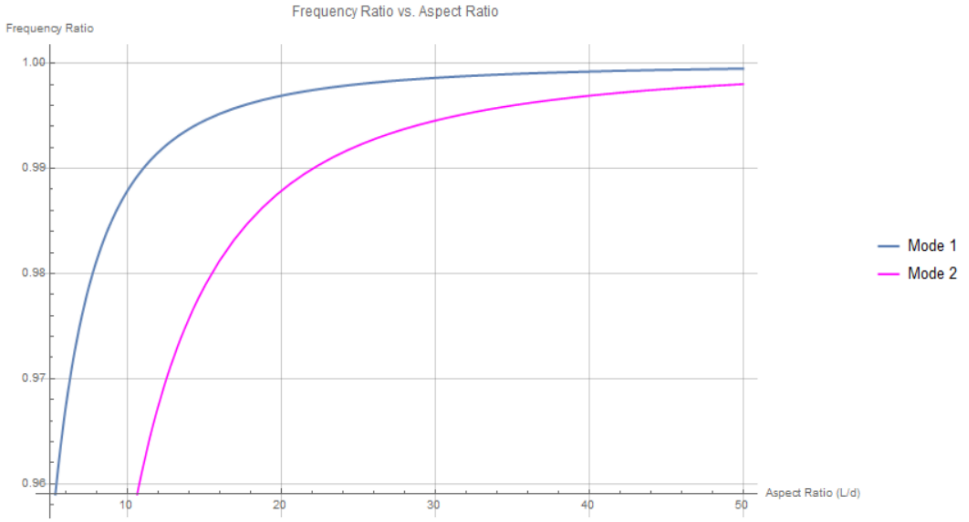


Figure 7. Effect of the aspect/slenderness ratio of the nanobeam (L/d) on the frequency ratio at a constant nonlocal parameter.

Analysis of Figure 7 reveals the following important points:

1. **Short and thick nanobeams ($L/d \leq 10$):** For smaller aspect ratios, the deviation from classical behavior is highly significant, and the nonlocal effects and structural discontinuity completely dominate the dynamic response.
2. **Long and slender nanobeams ($L/d \geq 30$):** With an increase in the aspect ratio, the frequency ratio approaches an asymptotic limit of 1.0. This behavior indicates that in very long and slender beams, small-scale effects diminish, and classical elasticity theory achieves sufficient accuracy for engineering analyses.

6. Conclusion

In this paper, the longitudinal free vibrations of single-walled carbon nanobeams based on Eringen's nonlocal elasticity theory were investigated using both an exact analytical method and a numerical finite element simulation. The governing differential equation of the system was derived, and following its analytical solution in Mathematica, a three-dimensional finite element model was developed in COMSOL Multiphysics for validation. The key findings of this research are summarized as follows:

1. **Model Validation:** The fundamental natural frequency calculated from the analytical solution showed good agreement with the 3D COMSOL simulation result. A relative error of less than 1% confirms the reliability of the analytical formulation and the accuracy of the finite element mesh.
2. **Stiffness Softening Phenomenon:** The presence of the nonlocal scale parameter ($e_0 a$) causes a reduction in the natural frequencies of the nanobeam across all modes. This behavior occurs due to the inclusion of interatomic bonding effects and the reduction of equivalent structural stiffness at the nanoscale.
3. **Sensitivity of Higher Vibration Modes:** The magnitude of frequency drop is strongly dependent on the vibration mode number. With an increase in the nonlocal parameter from 0 nm to 3 nm, the frequency ratio decreases by about 10% for the first mode, 25% for the second mode, and more than 40% for the third mode, indicating the high sensitivity of shorter wavelengths to nonlocal effects.
4. **Effect of Aspect Ratio (L/d):** The slenderness ratio analysis showed that for short nanobeams ($L/d \leq 10$), small-scale effects completely dominate; however, with a progressive increase in the length-to-diameter ratio ($L/d \geq 30$), the behavior of the nanobeam approaches the classical elasticity state, and the frequency ratio asymptotically approaches unity (1.0).

The results of this paper can serve as an accurate reference for the design and dynamic analysis of nanoelectromechanical resonators (NEMS), nano-frequency sensors, and longitudinal displacement transmission systems in nanorobotics.

Limitations and Future Work

Despite the accuracy and efficiency of the proposed analytical-3D FEA framework, several limitations of the current study should be acknowledged. First, the formulation and validation are primarily restricted to ideal clamped-clamped boundary conditions. Second, the model relies on continuum nonlocal elasticity theory without direct comparison against molecular dynamics (MD)

simulations or experimental nano-testing data. Future extensions of this research will focus on incorporating general non-rigid elastic boundaries, accounting for thermal-structural coupling effects, and performing comprehensive validation against experimental or atomistic benchmarks.

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